Manifolds, V2016

Problem sheet 1, to be discussed on Monday the 1st February 2016.

Problem 1. Show that $\mathbb{R}^n \setminus \{0\}$ is diffeomorphic to $S^{n-1} \times \mathbb{R}$ for every positive integer n.

Problem 2. Let m, n be integers such that $m \geq 0$, $n > 0$, and let $V \subset \mathbb{R}^{m+n}$ be an m-dimensional linear subspace. Show that $\mathbb{R}^{m+n} \setminus V$ is diffeomorphic to $S^{n-1} \times \mathbb{R}^{m+1}$.

Problem 3. Referring to Problem 1-7 in Lee's book, show that stereographic projection

$$S^n \setminus \{N\} \to \mathbb{R}^n$$

is a diffeomorphism.

Problem 4. The *Hopf map* $f : S^3 \to S^2$ can be defined as follows. Regarding S^3 as the unit sphere in $\mathbb{C}^2$ and S^2 as the unit sphere in $\mathbb{R} \times \mathbb{C}$ set

$$f(w, z) := (w\bar{w} - z\bar{z}, 2\bar{w}z).$$

(i) Show that f is well-defined and smooth.

(ii) Show that $f(w_1, z_1) = f(w_2, z_2)$ if and only if there exists a complex number a of norm 1 such that $(w_2, z_2) = (aw_1, az_1)$.

(iii) Draw a picture of the subset $f^{-1}(1, 0) \cup f^{-1}(-1, 0)$ of S^3, using stereographic projection.