Manifolds, Spring 2016: Assignment

To be returned no later than Thursday the 14th April at 14:30.

Problem 1. Compute the Lie bracket $[V, W]$ of the following vector fields on $\mathbb{R}^2$:

$$V = y^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$
$$W = x \frac{\partial}{\partial y}.$$

Problem 2. Compute the flow of the following vector field on $\mathbb{R}^2$:

$$V = y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Problem 3. Let $G \subset M(3 \times 3, \mathbb{R})$ be the space of all matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{R}$.

(i) Show that G is an embedded submanifold of $M(3 \times 3, \mathbb{R})$ diffeomorphic to $\mathbb{R}^3$.

(ii) Show that G is a group under matrix multiplication, and that this makes G a Lie group.

(iii) Find a basis v_1, v_2, v_3 for the tangent space $T_I G$, where I is the identity matrix.

(iv) For $i < j$ express $[v_i, v_j]$ as a linear combination of v_1, v_2, v_3. (Here, $T_I G$ has the Lie algebra structure induced from the Lie algebra of G.)

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Problem 4. Set

$$L := \{(x, y) \in \mathbb{R}^2 \,:\, x^3 = y^5\}.$$

Show that $L \setminus \{(0, 0)\}$ is an embedded submanifold of $\mathbb{R}^2$, but that L is not.

Problem 5. **(i)** Show that the map

$$F : \mathbb{R} \times \mathbb{C} \to \mathbb{C}^2, \quad (t, z) \mapsto (z^2, tz).$$

restricts to an immersion $f : S^2 \to \mathbb{C}^2$, where

$$S^2 := \{(t, z) \in \mathbb{R} \times \mathbb{C} \,:\, t^2 + |z|^2 = 1\}.$$

(ii) Let $\pi : S^2 \to \mathbb{RP}^2$ be the projection. Show that there exists a unique map $g : \mathbb{RP}^2 \to \mathbb{C}^2$ such that

$$f = g \circ \pi,$$

and that g is a smooth embedding.