First obligatory exercise, Mek3100/4100 2005

Preliminary version

March 2, 2005

Deadline and rules

You must turn in your paper before 15.00, Monday, 7'th March at "Ekspedisjonskontoret", Mathematics Institute (7th floor NHAbelshus). As experience shows that there are long queues around deadline, you are adviced to turn in your paper early. Remember to put your name and the number of your group/name of your program on your paper! If you due to illness or other circumstances have a need to extend the deadline, you must apply for an extension to Elisabeth Seland (room B718, NHAbelshus, e-mail:studieinfo@math.uio.no, phone 22 85 59 07). Remember that illness has to be documented by a medical doctor.
See http://www.math.uio.no/academics/obligregler.shtml for more information about rules concerning compulsory assignments at the Mathematical Institute. The assignment is compulsory, and students who do not get their paper accepted, will not get access to the final exam. To get the paper accepted, you must answer part 1,2 and 3 completely and present at least the framework, including the explicit steps in the perturbation expansion, for part 4. As you may get partial credit for a problem even without having reached a final answer, it is important that you turn in all your work. In solving the problems you may collaborate with others and use all tools avaliable. However, the paper ypu turn in must be written by you personally (by hand or computer) and it should reflect your understanding of the material. If we are not certain that you understand your own paper, we may ask you to give an oral presentation.

1 Helmholtz method

A differential equation of second order is defined for $0 \leq x \leq L$ according to

$$(p(x)y')' + q(x)y = 0 \tag{1}$$

where p is always positive. In addition we have boundary conditions that will be specified later. We know a solution y_0 of (1) and the boundary conditions. Moreover we define the corresponding inhomogeneous problem

$$(p(x)y')' + q(x)y = F(x), \tag{2}$$

with the same boundary conditions as the homogeneous one. In some cases We will show that a solution of (2) and the boundary conditions cannot exist unless (integration by parts must be used)

$$\int_0^L F y_0 \mathrm{d}x = 0 \tag{3}$$

Cases where the criterion (3) shall be proved

1.1 Periodic case

All of y_0, p and q are periodic, with period L. The inhomogeneous solution y shall also be periodic with period L.

1.2 Homogeneous boundary conditions

The boundary conditions are $y(0) = y(L) = 0$. We also assume that there are a non-trivial homogeneous solution that fulfill the same boundary conditions $y_0(0) = y_0(L)$. The latter generally require that we actually are dealing with an eigenvalue problem.

2 Eigenoscillations in a basin

Long waves in a shallow, 2D, basin with impermeable sidewalls are governed by the (non-dimensional) partial differential equation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(h(x)\frac{\partial \eta}{\partial x}\right) = 0 \quad \text{for} \quad 0 < x < 1, \tag{4}$$

and the boundary conditions

$$\frac{\partial \eta}{\partial x} = 0 \quad \text{for} \quad x = 0, 1. \tag{5}$$

η and h denotes the surface elevation and equilibrium depth, respectively. We are seeking periodic solutions of the form

$$\eta = z(x)\cos\omega t,$$

where z and ω must be determined.

2.1 The eigenvalue problem

Show that the periodic solution is governed by the ordinary differential equation

$$(hz')' + \omega^2 z = 0, \quad z(0) = z(L) = 0, \tag{6}$$

where $'$ denotes differentiation with respect to x.

2.2 Flat bottom

We first assume that the depth is constant: $h \equiv 1$. Find all solutions of (6).

2.3 Variable bottom

We assume a small perturbation of the bottom $h = 1 + \epsilon g(x)$. Explain how a perturbation solution of (6) may be found by Poincare' – Lindstedts method. Work out the solution, including $O(\epsilon)$, for

(i) $g(x) = \sin(kx)$

(ii) $g(x) = x$

Do this both by using Helmholtz method and by direct solution of the equations to order ϵ.

2.4 Test of perturbation solution

We now employ a constant bottom perturbation $g(x) = 1$. Find the perturbation solution and verify it by comparison with the exact solution.

3 Quadratic nonlinearity

Redo exercise 10 in the leaflet. Find y_0, y_1, ω_0, ω_1, ω_2. The last may be found by application of Helmholtz method.

4 Resistance and weak resonant forcing

For large Reynold's numbers the motion of a pendulum with fluid resistance may be governed by the equation

$$x'' + \epsilon x'|x'| + x = \epsilon F \cos t, \tag{7}$$

where ϵ is small. Use Helmholtz method to find a periodic response.