ECON3120/4120 Mathematics 2: postponed exam 2025-01-15

(4 hours. Support material: "Rules and formulas" attachment and Inspera's simple calculator.)

For the entire problem set:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Suggested weighting: Problem 4 and each letter-enumerated problem ("(a)", "(b)", ...) count equal.

Problem 1 Let $\mathsf{A}_t = \begin{pmatrix} t & 0 & 0 & t \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix}$, $\mathsf{M} = \begin{pmatrix} 3 & 0 & 0 & -6 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 4 & 0 \\ -6 & 0 & 0 & 6 \end{pmatrix}$ and $\mathsf{b} = \begin{pmatrix} 0 \\ -1 \\ 2 \\ -3 \end{pmatrix}$, for each real constant t. Throughout this problem, the prime symbol ( ) denotes matrix transpose.

(a) Calculate the matrix product(s) that is/are well-defined among the following, and identify any which isn't/aren't well-defined:

i) $\mathsf{b}\mathsf{b}$ ii) $\mathsf{b}\ \mathsf{b}$ iii) $\mathsf{M}\mathsf{b}$ iv) $\mathsf{M}\,\mathsf{A}_t$

(b) There is some number s and some number t such that $(s\mathsf{A}_{t})^{-1} = \mathsf{M}$.

Find s and t .

(c) For each of A_t, M and b: Calculate the determinant, or identify it as not well-defined.

(d) In this part, consider equation systems $\mathsf{A}_t\mathsf{x} = \mathsf{b}$ (where x is the unknown):

- Solve the system for $t = t$ found in part (b). You are allowed to express the solution in terms of t and the number s (also from (b)) whether or not you found those numbers, but your solution must be calculated out.
- Find some $t = t_0$ such that no solution exists.

Problem 2 Let $f(x) = 1 - e^{x^{1/3}}$ be defined for all real x. Note, $x^{1/3}$ has the same sign as x.

(a) Show that $\lim_{x \to +\infty} \frac{f(x)}{1+x} = -\infty$ and find $\lim_{x \to -\infty} \frac{f(x)}{1+x}$.

For each constant $c > 0$, let $h(x) = f(x) + c \cdot (1 + x)$. Note that $h(0) = 1 + c$.

(b) Show that h has a zero $x_1 < 0$ and a zero $x_2 > 0$.

(You are not required to find them, nor to take a stand on whether there are more.)

(c) Each of the zeroes x_1, x_2 depends on c. Find expressions for $\frac{dx_1}{dc}$ and $\frac{dx_2}{dc}$.

Problem 3 Let $Q(x, y) = x^{1/4}y^{1/2}$, let $R(x, z) = x^{1/4}z^{1/2}$ and for each constant a $(0, 1)$, let $f(x, y, z) = 4(1 - a) \cdot Q(x, y) + 4a \cdot R(x, z)$.

(a) Which of the functions Q, R and f is/are homogeneous?

Let now p, w and b be positive constants, and consider the problem

$$\max \ f(x, y, z) \qquad \text{subject to} \quad x + py + wz = b \tag{P}$$

Observe that any admissible point with $x = 0$ is «worst possible», it yields $f = 0$.

(b) Can we tell, without looking into the Lagrange conditions, whether there must be a point that satisfies them?

Hint: Careful about the logic!

(c) Show that if $a = \dfrac{w}{p + w}$ (so that $1 - a = \dfrac{p}{p + w}$), then an optimal point must have $y = z$.

(Recall that x must be $= 0$ in optimum.)

(d) Change a from $\dfrac{w}{p + w}$ to $\dfrac{w}{p + w} + h$, and let ΔV be the change in optimal value.

Find $\lim\limits_{h \to 0} \dfrac{\Delta V}{h}$.

Problem 4 *(expected to count as one letter-enumerated item)*

Let $H > 0$ be a constant. Suppose that the state variable evolves according to $x_{t+1} = x_t \cdot (1 - c_t)$ and consider the dynamic programming problem

$$v_{t_0}(x_{t_0}) = \max_{c_t \in (0,1)} \left\{ 2\sqrt{c_{t_0} x_{t_0}} + 2\sqrt{c_{t_0+1} x_{t_0+1}} + \ldots + 2\sqrt{c_{T-1} x_{T-1}} \quad + \quad 2H\sqrt{x_T} \right\}$$

It is a fact that v_{T-1} is of the form $h_1 \cdot \sqrt{x_{T-1}}$ for some positive constant h_1.

Show that v_{T-2} is of the form $h_2 \cdot \sqrt{x_{T-2}}$ for some positive constant h_2.

(You do not have to calculate h_2 as long as you establish that it is a positive constant.)

Problem 5

(a) Among the following four integrals, two are «doable by hand in reasonable time»:

i) $\int_0^1 y^{2024} \cdot e^y \, dy$ ii) $\int_1^0 y \cdot (e^y)^{2024} \, dy$ iii) $\int y \cdot (\ln y)^{2024} \, dy$ iv) $\int y^{2024} \cdot \ln y \, dy$

It is part of your task to identify which ones can be done reasonably fast:

- Pick one of the integrals and calculate it.
- Explain which of the other three takes fewer operations and less time than the two remaining, *and why.*

(b) Consider the differential equation $\dot{x} + 2tx = -t$.

- Find a *constant* particular solution $x(t) = \overline{p}$.
- Find the general solution.

(End of problem set. Attachment: Rules and formulas.)