ECON3120/4120 Mathematics 2 2024: on the 2025-05-15 exam

See the guideline for the December exam. Feedback from an external grader was that that problem set was too long and too easy – but the length dominating, thus the grading scale was adjusted downwards. Although this set was intended to be about the same overall – indicating, the same grading scale could be used – it was intended to be a bit shorter and not so easy. Some items mentioned:

- Problem 1. Linear algebra typically scores higher than average, and with 4 parts it carries more weight than at the ordinary exam. However, part (b) is a question type that this year seems not to be well understood: you got a solution candidate – and just verify that the equation(s) hold true. That could be done very quickly, but it is not to say that it is in practice much easier than a problem set average.
- Problem 2. $\exp(x^{1/3})$ is not an exponential function of x. Anticipated: weaker candidates will claim it is – on top of having to distinguish between growth and decay.

  As goes problem 2(c), then one would expect this to be *a lot* easier than finding derivatives when two functions are determined by two equations (cf. the bad scores on ordinary exam part 1(b)); after all, this is «Math 1», slope of level curves. However, experience is that such problems do not score well on a Math 2 exam. It is definitely quicker done, to those who know it.

Considering Problems 1 and 2 part (c) compared with (equal weight) ordinary exam's problems 1 and 2, the likely single factor that makes di�erence in di�culty, is the absence of ordinary set's problem 1(a), which was easy and scored well. Then on the other hand, the new 1 and 2c should be possible to do faster despite having two 4×4 matrix products.

- Problem 3:
  - Part (b) will likely reveal those candidates who merely namedrop the extreme value theorem without checking its applicability. On the contrary: Evidence from the ordinary exam indicates that the candidates do not know what «Lagrange conditions» are, and so a badly answered question was removed.
  - Part (d) requires not only the knowledge of the envelope theorem, but also the Math 1 knowledge of *what the derivative is.*
- Problem 4: This is one of the prototypical dynamic programming problems covered in class, and about as short as such a problem could get. Although it requires something in the vein of the ordinary exam's Problem 4(b) (which scored lesser than (a)), the overall weight on dynamic programming is now only half of at the ordinary exam, and so

Overall, despite Problem 3(a) being easy, problems 3 and 4 might be harder than on the ordinary exam. Problem 5 is arguably easier (but likely not so easy as when 2ab are factored in).

- Problem 5: Part (a) has the prototypical integration by parts problems: polynomial times exp, and power times log. Unlike the ordinary, there is no $(\ln z)^2$. If they do not know that $y^N e^(ay)$ requires integration by parts, and more if N is big than if $N = 1$, then they haven't learned the basics of the method. The «explain» part takes very little time to those who know; those who will have to attempt the calculations, might need more time, but this isn't witchcraft.

Problem 5 part (b): The ordinary exam asked to find particular solutions of the form $q2^{-t} + h$ to both a difference equation and a differential equation. Here, there is only one and $q = 0$. Then the ordinary exam had a question on solving an initial value problem, whereas this one asks for a general solution.
The «hard» part of this one is that they «have to» do what they are asked. Merely trusting the formula will lead to an integral that requries substitution, which is no longer taught. However, getting out a particular solution – or just calling it «$\bar{p}$»! – will get out $Ce^{t^2} + \bar{p}$ immediately.

To follow: Problems restated with solution sketches.

Problem 1 Let $A_t = \begin{pmatrix} t & 0 & 0 & t \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 3 & 0 & 0 & -6 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 4 & 0 \\ -6 & 0 & 0 & 6 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ -1 \\ 2 \\ -3 \end{pmatrix}$, for each real constant t. Throughout this problem, the prime symbol ( ) denotes matrix transpose.

(a) Calculate the matrix product(s) that is/are well-defined among the following, and identify any which isn't/aren't well-defined:

i) bb ii) $b\ b$ iii) Mb iv) $M A_t$

(b) There is some number s and some number t such that $(sA_{t})^{-1} = M$.

Find s and t .

(c) For each of A_t, M and b: Calculate the determinant, or identify it as not well-defined.

(d) In this part, consider equation systems $A_t x = b$ (where x is the unknown):

- Solve the system for $t = t$ found in part (b). You are allowed to express the solution in terms of t and the number s (also from (b)) whether or not you found those numbers, but your solution must be calculated out.
- Find some $t = t_0$ such that no solution exists.

Solution sketches with remarks:

(a) bb undefined, since b not square.

$b\ b = 0 + (-1)^2 + 2^2 + (-3)^2 = \underline{\underline{14}}$ (or strictly speaking «(14)» but we don't care to distinguish a one-by-one from its element.)

$$Mb = \begin{pmatrix} 0 + 0 + 0 + (-6)\cdot(-3) \\ 0 + (-2)\cdot(-1) + (-2)\cdot 2 + 0 \\ 0 + (-2)\cdot(-1) + 4\cdot 2 + 0 \\ 0 + 0 + 0 + 6\cdot(-3) \end{pmatrix} = \underline{\underline{\begin{pmatrix} 18 \\ -2 \\ 10 \\ -18 \end{pmatrix}}}$$

$$MA_t = \begin{pmatrix} 3 & 0 & 0 & -6 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 4 & 0 \\ -6 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} t & 0 & 0 & t \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ t & 0 & 0 & 1 \end{pmatrix} \text{ (in that order!), equals}$$

$$\begin{pmatrix} 3\cdot t+0+0+(-6)\cdot t & 0+0+0+0 & 0+0+0+0 & 3t+0+0+(-6) \\ 0+0+0+0 & 0+(-4)+(-2)+0 & 0-2+2+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0-2-4+0 & 0+0+0+0 \\ -6t+0+0+6t & 0+0+0+0 & 0+0+0+0 & -6t+0+0+6 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -3t & 0 & 0 & 3t-6 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6t+6 \end{pmatrix}}}$$

(b) It suffices to find s and t such that $sI = MA_{t_0}$. Put $\underline{\underline{t = 2}}$ to get a diagonal matrix; then observe that $MA_2 = -6I$, so $\underline{\underline{s = -6}}$.

(c) b has no determinant, it is not square.

Cofactor expansion along the first row and then along the last row, will yield:

$|M| = 3\begin{vmatrix} -2 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix} - (-6)\begin{vmatrix} 0 & -2 & -2 \\ 0 & -2 & 4 \\ -6 & 0 & 0 \end{vmatrix} = 18\begin{vmatrix} -2 & -2 \\ -2 & 4 \end{vmatrix} - 36\begin{vmatrix} -2 & -2 \\ -2 & 4 \end{vmatrix}$, same 2×2 determinants which do equal $-8-4 = -12$ leading us to $-18\cdot(-12)\ \underline{\underline{216}}$

For $|A_t|$ we could do likewise, but it does not hurt to use that $|MA_t|$ equals $3t\cdot 36\cdot(6t-6)$ but at the same time $= |M|\,|A_t| = (18\cdot 12)|A_t|$ so $|A_t| = \frac{3t\cdot 36\cdot(6t-6)}{18\cdot 12} = \underline{\underline{3t(t-1)}}$.

(d) $A_t x = b$:

- The problem text of part (b) tells us that s and t exist, and using that we can get as far as $A_{t_0} = \frac{1}{s}M$, so $x = \frac{1}{s}Mb = \frac{1}{s}\left(18,\ -2,\ 10,\ -18\right)^\top$ from part (a).

  With $s = -6$ as calculated: $\frac{-1}{6}\begin{pmatrix} 18 \\ -2 \\ 10 \\ -18 \end{pmatrix} = \underline{\underline{\frac{1}{3}\begin{pmatrix} -9 \\ 1 \\ -5 \\ 9 \end{pmatrix}}}$ (or if you like, $\begin{pmatrix} -3 \\ 1/3 \\ -5/3 \\ 3 \end{pmatrix}$).

- t_0 must necessarily make the determinant vanish, so from (a) it must be either 0 or 1. $t = 0$ yields that the first equation says $0 = 0$ and is superfluous, so we might have better hope[a] trying $t = 1$. In that case, the first row becomes $(1\ 0\ 0\ 1\ |\ 0)$ and the last row becomes $(1\ 0\ 0\ 1\ |\ -3)$ leading to the contradiction $0 = -3$. Thus, $\underline{\underline{\text{no solution for } t = t_0 = 1}}$.

[a] and it so happens that $t = 0$ yields several solutions: x_1 does not enter and is free, x_4 only enters in the fourth equation $x_4 = -3$, and the equation for $(x_2, x_3)^\top$ has invertible coefficient matrix.

Problem 2 Let $f(x) = 1 - e^{x^{1/3}}$ be defined for all real x. Note, $x^{1/3}$ has the same sign as x.

(a) Show that $\lim_{x\to+\infty} \frac{f(x)}{1+x} = -\infty$ and find $\lim_{x\to-\infty} \frac{f(x)}{1+x}$.

For each constant $c > 0$, let $h(x) = f(x) + c\cdot(1+x)$. Note that $h(0) = 1 + c$.

(b) Show that h has a zero $x_1 < 0$ and a zero $x_2 > 0$.

(You are not required to find them, nor to take a stand on whether there are more.)

(c) Each of the zeroes x_1, x_2 depends on c. Find expressions for $\frac{dx_1}{dc}$ and $\frac{dx_2}{dc}$.

Solution sketches with remarks:

(a) $x \to -\infty$ leads to $\exp(x^{-1/3}) \to e^{-\infty} = 0$, so $\lim_{x\to-\infty} \frac{f(x)}{1+x} = \frac{1-0}{-\infty} = \underline{\underline{0}}$.

For the limit as $x \to +\infty$, we can of course let $x^{1/3} = y$ and consider $\lim_{y\to+\infty} \frac{1-e^y}{1+y^3} = 0 - \lim_{y\to+\infty} \frac{e^y}{1+y^3} = 0 - \infty$ now we got an exponential numerator and a polynomial denominator. But sans that trick: fraction $\to$ $-$«∞/∞» and l'Hôpital's rule yields $0 - \lim_{x\to+\infty} \frac{e^{x^{1/3}}\cdot\frac{1}{3}x^{-2/3}}{1} = \frac{-1}{3}\lim_{x\to+\infty}\frac{e^{x^{1/3}}}{x^{2/3}}$, again infinity over infinity. l'Hôpital again: $\frac{-1}{3}\lim_{x\to+\infty}\frac{e^{x^{1/3}}\cdot\frac{1}{3}x^{-2/3}}{\frac{2}{3}x^{-1/3}}$ which $= \frac{-1}{6}\lim_{x\to+\infty}\frac{e^{x^{1/3}}}{x^{1/3}}$, and again infinity over infinity. Now both numerator and denominator depend solely on $x^{1/3}$, and its derivative $\frac{1}{3}x^{-2/3}$ cancels:

$$\frac{-1}{6}\lim_{x\to+\infty}\frac{e^{x^{1/3}}\cdot\frac{1}{3}x^{-2/3}}{1\cdot\frac{1}{3}x^{-2/3}} = \frac{-1}{6}\lim_{x\to+\infty} e^{x^{1/3}} = -\infty.$$

(b) $h(0) = 1 + c > 0$. h is continuous, and if we can show that $h < 0$ *somewhere* in $(-\infty, 0)$ then the intermediate value theorem will ensure it hits zero for some $x_1 < 0$; and, if $h < 0$ *somewhere* in $(0, \infty)$ then the intermediate value theorem will ensure it hits zero for some $x_2 > 0$.

- As $x \to -\infty$, $f(x) \to 1 - 0 = 1$ and $h(x) \to 1 + c + c\cdot(-\infty) = -\infty$ since $c > 0$. Therefore, h becomes negative for some $x < 0$, and x_1 exists.
- Consider $\lim_{x\to+\infty}\frac{h(x)}{1+x} = c + \lim_{x\to+\infty}\frac{f(x)}{1+x} = -\infty$, so h becomes negative for some $x > 0$, thus x_2 exists.

(c) For $i = 1, 2$ we have $f(x_i) + (1+x_i)\cdot c = 0$. Total derivative wrt. c: $(1+x_i) + \left(f'(x_i) + c\right)\cdot\frac{dx_i}{dc} = 0$, so for both x_1 and x_2 we have the expression

$$\frac{dx_i}{dc} = -\frac{1+x_i}{c + f(x_i)} = \underline{\underline{-\frac{1+x_i}{c - \frac{1}{3}x_i^{-2/3}\exp(x_i^{1/3})}}}$$

(Of course it is possible to replace $e^{x^{1/3}}$ by $1 + (1+x_i)c$.)

Problem 3 Let $Q(x, y) = x^{1/4}y^{1/2}$, let $R(x, z) = x^{1/4}z^{1/2}$ and for each constant $a \in (0, 1)$, let $f(x, y, z) = 4(1 - a) \cdot Q(x, y) + 4a \cdot R(x, z)$.

(a) Which of the functions Q, R and f is/are homogeneous?

Let now p, w and b be positive constants, and consider the problem

$$\max \ f(x, y, z) \qquad \text{subject to} \quad x + py + wz = b \tag{P}$$

Observe that any admissible point with $x = 0$ is «worst possible», it yields $f = 0$.

(b) Can we tell, without looking into the Lagrange conditions, whether there must be a point that satisfies them?

Hint: Careful about the logic!

(c) Show that if $a = \dfrac{w}{p + w}$ (so that $1 - a = \dfrac{p}{p + w}$), then an optimal point must have $y = z$.

(Recall that x must be $= 0$ in optimum.)

(d) Change a from $\dfrac{w}{p + w}$ to $\dfrac{w}{p + w} + h$, and let ΔV be the change in optimal value.

Find $\lim_{h \to 0} \dfrac{\Delta V}{h}$.

Solution sketches with remarks:

(a) $Q(tx, ty) = t^{1/4}x^{1/4}t^{1/2}y^{1/2} = t^{3/4}Q(x, y)$.

$R(tx, tz) = t^{1/2}x^{1/2}t^{1/4}z^{1/4} = t^{3/4}R(x, z)$.

$f(tx, ty, tz) = 4(1 - a)t^{3/4}Q(x, y) + 4at^{3/4}R(x, z) = t^{3/4}f(x, y, z)$.

So all three are homogeneous (of the same degree 3/4).

(b) Remark: This is a difficult question to get to a *strictly correct* level, especially since we are in three dimensions, and depending on argument: both an affirmative and a negative answer could retrieve «at least very high» scores, the latter if they point out that a plane is not bounded. Note that the extreme value theorem does not apply without considering the domain of f! Here is an argument that does consider it:

f is only defined where each variable is nonnegative. Because the coefficients 1, p, w are all positive, we are restricted to a bounded set: x cannot be outside $[0, b]$, y must be $\in [0, b/p]$ and $z \in [0, b/z]$. The set is closed (the boundary is included, each variable can be zero) and the function is continuous. The extreme value theorem grants existence.

(c) Lagrange conditions (note: they are taken as necessary in Math 2) with $a = w/(p + w)$:

$$x + py + wz = b \tag{1}$$
$$\frac{p}{p+w}x^{-3/4}y^{1/2} + \frac{w}{p+w}x^{-3/4}z^{1/2} = \lambda \tag{2}$$
$$\frac{p}{p+w}\cdot 2x^{1/4}y^{-1/2} = p\lambda \tag{3}$$
$$\frac{w}{p+w}\cdot 2x^{1/4}z^{-1/2} = w\lambda \tag{4}$$

Eliminate λ from (3) and w from (4), using that $pw = 0$:

$$\frac{1}{p+w}\cdot 2x^{1/4}y^{-1/2} = \lambda = \frac{1}{p+w}\cdot 2x^{1/4}z^{-1/2} \tag{5}$$

Then $x^{1/4}y^{-1/2} = x^{1/4}z^{-1/2}$, and since $x = 0$ (in optimum – as noted, $x = 0$ yields worst possible outcome $f = 0$), then $y^{1/2}$ must equal $z^{1/2}$ and thus $y = z$.

(d) Observe first that what is asked for, is the derivative wrt. a. By the envelope theorem we can take the *partial* derivative first, and insert afterwards. Note that $y = z$ can be invoked without having attempted at solving part (c), as it was given in the problem text: $\frac{\partial}{\partial a}\left[4(1-a)x^{1/4}y^{1/2} + 4ax^{1/4}z^{1/2} - \lambda(x + py + wz - b)\right] = -4x^{1/4}y^{1/2} + 4x^{1/4}z^{1/2} = 4x^{1/4}(z^{1/2} - y^{1/2})$ is <u>zero</u> in optimum.

Problem 4 *(expected to count as one letter-enumerated item)*

Let $H > 0$ be a constant. Suppose that the state variable evolves according to $x_{t+1} = x_t\cdot(1 - c_t)$ and consider the dynamic programming problem

$$v_{t_0}(x_{t_0}) = \max_{c_t\ (0,1)}\left\{2\sqrt{c_{t_0}x_{t_0}} + 2\sqrt{c_{t_0+1}x_{t_0+1}} + \ldots + 2\sqrt{c_{T-1}x_{T-1}} \quad + \quad 2H\sqrt{x_T}\right\}$$

It is a fact that v_{T-1} is of the form $h_1\cdot\sqrt{x_{T-1}}$ for some positive constant h_1.
Show that v_{T-2} is of the form $h_2\cdot\sqrt{x_{T-2}}$ for some positive constant h_2.
(You do not have to calculate h_2 as long as you establish that it is a positive constant.)

The following is a complete solution (since in this part of the course, they do not need to touch upon the existence of optimum):

By the dynamic programming equation, $v_{T-1}(x) = \max_{c\ (0,1)}\left\{2\sqrt{cx} + h_1\sqrt{x(1-c)}\right\}$ which since $\sqrt{x} \geq 0$, equals $\sqrt{x}\cdot\underbrace{\max_{c\ (0,1)}\left\{2\sqrt{c} + h_1\sqrt{1-c}\right\}}_{\text{call it } h_2}$. And $\underbrace{2\sqrt{c} + h_1\sqrt{1-c}}_{>0,\text{ sum of positives}}$ because $h_1 > 0$.

Problem 5

(a) Among the following four integrals, two are «doable by hand in reasonable time»:

i) $\int_0^1 y^{2024}\cdot e^y\,dy$ ii) $\int_1^0 y\cdot(e^y)^{2024}\,dy$ iii) $\int y\cdot(\ln y)^{2024}\,dy$ iv) $\int y^{2024}\cdot\ln y\,dy$

It is part of your task to identify which ones can be done reasonably fast:

- Pick one of the integrals and calculate it.
- Explain which of the other three takes fewer operations and less time than the two remaining, *and why.*

(b) Consider the differential equation $\dot{x} + 2tx = -t$.

- Find a *constant* particular solution $x(t) = \overline{p}$.
- Find the general solution.

Solution sketches with remarks:

(a) ii) requires only one integration by parts (differentiating y) while i) requires 2024 such operations. iii) requires only one integration by parts (differentiating $\ln y$) while iv) requires 2024 (successive differentiations of $(\ln y)^n$.)

Problem asks to do one of them. For this solution note, let's do both.

ii) (Alternatively, do indefinite first. There is a reversal there from $\int_a^b = -\int_b^a$):

$$\begin{aligned}\int_1^0 ye^{2024y}\,dy &= \Big[y\frac{e^{2024y}}{2024}\Big]_1^0 - \int_1^0 1\cdot\frac{e^{2024y}}{2024}\,dy\\ &= 0 - \frac{e^{2024}}{2024} + \Big[\frac{e^{2024y}}{(2024)^2}\Big]_0^1\\ &= e^{2024}\Big(\frac{1}{2024^2} - \frac{1}{2024}\Big) + \frac{1}{2024^2}\\ &= \underline{\underline{\frac{1}{2024^2}\Big(1 - 2023e^{2024}\Big)}}\end{aligned}$$

iii)

$$\begin{aligned}\int y^r\cdot\ln y\,dy &= \frac{1}{r+1}y^{r+1}\ln y - \int\frac{1}{r+1}y^{r+1}\cdot\frac{1}{y}\,dy\\ &= \frac{1}{r+1}y^{r+1}\ln y - \frac{1}{(r+1)^2}y^{r+1} + C\end{aligned}$$

holds for $r = -1$. With $r = 2024$ we get $\underline{\underline{C + \frac{y^{2025}}{2025}\Big(\ln y - \frac{1}{2025}\Big)}}$

(b) Constant particular: Insert $\overline{p}$ (whose derivative is 0) to get $0 + 2t\overline{p} = -t$ and $\underline{\underline{\overline{p} = -1/2}}$.

General solution: $(t^2) = 2t$ yields $\underline{\underline{Ce^{t^2} + \overline{p} = Ce^{t^2} - 1/2}}$.